18.100A PSET 5 SOLUTIONS

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General Comments

A lot of solutions were correct but significantly more complicated than what was necessary.

Problem 1

(a). Since I is a compact interval, the hypothesis implies that $f(I)$ is a compact interval, i.e., $f(I) = [c, d]$ for some real numbers $c \leq d$. But then $c \leq f(x)$ for all $x \in I$, and $d \geq f(x)$ for all $x \in I$. At the same time, $c \in [c, d] = f(I)$ and $d \in [c, d] = f(I)$, which by the definition of $f(I)$ means that the values c and d are achieved by f in the interval I. This means that c and d are the minimum and maximum, respectively, of f on the interval I.

(b). Let $I = [a, b]$. As $f(I) = [c, d]$ for some $c \leq d$ in the reals. Therefore, $c \leq f(a), f(b) \leq d$.

We now divide into two cases, depending on whether $f(a) \leq f(b)$ or $f(b) > f(a)$.

If $f(a) \leq f(b)$, then the interval $[f(a), f(b)]$ is contained in the interval $[c, d]$. But $[c, d] = f(I)$, so every value in $[f(a), f(b)]$ is attained.

If $f(b) < f(a)$, then the interval $[f(b), f(a)]$ is contained in the interval $[c, d]$. But $[c, d] = f(I)$, so every value in $[f(b), f(a)]$ is attained.

Problem 2

As the function is continuous on $[0, 1]$, we know it is uniformly continuous on that interval.

Now let $\epsilon > 0$, and let $\delta > 0$ such that $|\sqrt{x} - \sqrt{y}| < \epsilon$ whenever $|x - y| < \delta$ and $x, y \in [0, 1]$.

Now let $\delta' = \min(\delta, \epsilon)$.

We now show that for $x, y \in [0, \infty)$ such that $|x - y| < \delta'$, we have $|\sqrt{x} - \sqrt{y}| < \epsilon$. To show this, we divide into two cases.

If $x, y \in [0, 1]$, they we are done, because $|x - y| < \delta' \leq \delta$.

If one of x and y is greater than 1, then $\sqrt{x} + \sqrt{y} > 1$. We have the identity $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) = x - y$. Therefore,

$$|\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq |x - y| < \delta' \leq \epsilon.$$

As ϵ was arbitrary, and δ' is positive, we are done.

Problem 3

We fix $\epsilon > 0$. Let $\delta > 0$ be such that $\ln x < \epsilon$ for all $x \in [1, 1 + \delta]$.

If $x, y \in [1, \infty)$ such that $0 < x - y < \delta$, then

$$|\ln x - \ln y| = \ln x - \ln y = \ln x/y = \ln\left(1 + \frac{x - y}{y}\right).$$

But $y \geq 1$, so $\frac{x - y}{y} \leq x - y < \delta$. Therefore, $1 + \frac{x - y}{y} \in [1, 1 + \delta]$, so

$$|\ln x - \ln y| = \ln\left(1 + \frac{x - y}{y}\right) < \epsilon.$$

As ϵ was arbitrary, and δ is positive, we are done.

Problem 4

(a). We extend f to a continuous function on $[0, 1]$. We let $f(x) = x \sin 1/x$ for $x \in (0, 1]$, and we set $f(0) = 0$. It is clear that f is continuous for $x \in (0, 1]$, because $1/x$, sin, and x are continuous for such x.

At $x = 0$, we note that $-x \leq f(x) \leq x$ for $x > 0$, and $\lim_{x \to 0} x = \lim_{x \to 0} -x = 0$, so the Squeeze Theorem tells us that $\lim_{x \to 0^+} f(x) = 0$. This implies that f is continuous at $x = 0$.

Now the extension is uniformly continuous by Theorem 13.5, which implies that the original f, i.e., the restriction to $(0, 1)$, is also uniformly continuous.

(b). We prove uniform continuity. Let $\epsilon > 0$. We take $\delta = \epsilon$.

But $2|xy| \leq x^2 + y^2$, so

$$\begin{aligned} |xy - 1| &\leq 1 + |xy| \\ &\leq 1 + 2|xy| \\ &\leq 1 + x^2 + y^2 \\ &\leq 1 + x^2 + y^2 + x^2y^2 \\ &= (1 + x^2)(1 + y^2) \end{aligned}$$

It follows that $\left| \frac{1 - xy}{(1 + x^2)(1 + y^2)} \right| \leq 1$. Therefore, whenever $|x - y| < \delta$, we have

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{x}{1 + x^2} - \frac{y}{1 + y^2} \right| \\ &= \left| \frac{x + xy^2}{(1 + x^2)(1 + y^2)} - \frac{y + yx^2}{(1 + x^2)(1 + y^2)} \right| \\ &= \left| \frac{x + xy^2 - y - yx^2}{(1 + x^2)(1 + y^2)} \right| \\ &= \left| \frac{(x - y)(1 - xy)}{(1 + x^2)(1 + y^2)} \right| \\ &= |x - y| \left| \frac{1 - xy}{(1 + x^2)(1 + y^2)} \right| \\ &\leq |x - y| \\ &< \delta \\ &= \epsilon. \end{aligned}$$

This proves that $f(x)$ is uniformly continuous.

Problem 5

(a). Let $\epsilon > 0$. Let $\delta > 0$ such that $|f(x)-f(y)| < \epsilon$ whenever $|x-y| < \delta$ for $x, y \in I$ (which exists because f is uniformly continuous on I). Let $N > 0$ such that $|a_n - a_m| < \delta$ whenever $n, m > N$ (which exists because $\{a_n\}$ is Cauchy).

Then for $n, m > N$, we have $|a_n - a_m| < \delta$, so $|f(a_n) - f(a_m)| < \epsilon$. Since ϵ was arbitrary, we have shown that $\{f(a_n)\}$ is Cauchy.

(b). Let $a_n = \dfrac{1}{n}$ for integer $n \geq 1$. Then $\{a_n\}$ is a Cauchy sequence, because it converges to 0 in $\mathbb{R}$. Let $f(x) = 1/x$ on $I = (0, 1]$. If f were uniformly continuous on I, then $\{f(a_n)\} = \{n\}$ would be a Cauchy sequence. But this sequence is unbounded, a contradiction.

Problem 6

Suppose that f is uniformly continuous on (a, b), but that f is not bounded above. Then for any positive integer n, we can choose $a_n \in (a, b)$ such that $f(a_n) > n$. We make such a choice for each positive integer n.

Then $\lim_{n\to\infty} f(a_n) = n$. We choose a subsequence $\{b_n\}$ of $\{a_n\}$ such that $\{b_n\}$ converges to an element of $[a, b]$. It follows that $\{b_n\}$ is a Cauchy sequence in (a, b), so by part (a) of problem 5, we find that $\{f(b_n)\}$ must be a Cauchy sequence. But $\{f(b_n)\}$ is a subsequence of $\{f(a_n)\}$, so $\lim_{n\to\infty} f(b_n) = \infty$, which contradicts the claim that $\{f(b_n)\}$ is a Cauchy sequence. This contradiction proves the result.

Problem 7

Let $f(x) = \sin x$. We must find n for which our error

$$|R_n(x)| = |\sin x - T_n(x)|$$

is less than $1/1000$ for $|x| < 0.5$. We have

$$|R_n(x)| = \left| \frac{f^{(n+1)}(c)}{n!} x^{n+1} \right|$$

for some $c \in (-0.5, 0.5)$.

For all n, we know that $f^{(n+1)}$ is $\pm\sin$ or $\pm\cos$, so $|f^{(n+1)}(c)| \leq 1$. Therefore,

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c)}{n!}x^{n+1}\right| \leq \frac{|x|^{n+1}}{(n+1)!} < \frac{1}{2^n(n+1)!}.$$

For $n = 4$, we have $2^n(n+1)! = 2^4(5!) = (16)(120) = 1920 > 1000$, so the error is small enough for $n = 4$.

In fact, $T_3(x) = T_4(x)$ in this case, so you only need the first three.

We also note that $n < 3$ does not work. For $x = 0.4$, we have $\sin x = 0.38941\cdots$. Now $T_1(x) = T_2(x) = x$, and for $x = 0.4$, this clearly does not agree in the first four decimal places.

Problem 8

(a). Let $f(x) = \sin x$. We have

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c)}{n!}x^{n+1}\right|$$

for some $c \in \mathbb{R}$.

For all n, we know that $f^{(n+1)}$ is $\pm\sin$ or $\pm\cos$, so $|f^{(n+1)}(c)| \leq 1$. Therefore,

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c)}{n!}x^{n+1}\right| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

It suffices to show that $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$ for all n. But this was Problem 10 of the first problem set.

(b). We let $f(x) = \frac{1}{1-x}$. We first prove:

Lemma. *We have $f^{(n)}(x) = n!(1-x)^{-1-n}$ for all n.*

Proof. We use induction on n, starting at $n = 1$.

For $n = 1$, this is because $f'(x) = (1-x)^{-2}$.

Now suppose it is true for $n \geq 1$. Then $f^{(n+1)}(x) = (f^{(n)})'(x) = \frac{d}{dx}\left(n!(1-x)^{-1-n}\right) = n!(-1)(-1-n)(1-x)^{-2-n} = (n+1)!(1-x)^{1-(n+1)}$ by the lemma, as desired. □

For $x \in (-1, 0]$, we have

$$|R_n(x)| \leq \left|\frac{f^{(n+1)}(c)}{n!}x^{n+1}\right| = |(1-c)^{-2-n}x^{n+1}|$$

for some $c \in (x, 0]$. But $1-c \geq 1$, so $|R_n(x)| \leq= |(1-c)^{-2-n}x^{n+1}| \leq |x|^{n+1}$. Since $|x| < 1$, we find that this approaches 0 as $n \to \infty$.

Problem 9

Letting $f(x) = x^3 - 2x + 1$, we compute

$$f(x) = x^3 - 2x + 1$$

$$f(-1) = 2$$

$$f'(x) = 3x^2 - 2$$

$$f'(x) = 1$$

$$f''(x) = 6x$$

$$f''(-1) = -6$$

$$f'''(x) = 6$$

$$f'''(-1) = 6$$

$$f^{(4)}(x) = 0$$

$$f^{(4)}(-1) = 0$$

Taylor's formula then tells us that

$$T_4(x) = f(-1)+f'(-1)(x+1)+\frac{f''(-1)}{2}(x+1)^2+\frac{f'''(-1)}{6}(x+1)^3+\frac{f^{(4)}(-1)}{24}(x+1)^4.$$

Plugging in our values, we get

$$T_4(x) = 2 + (x+1) - 3(x+1)^2 + (x+1)^3.$$

Problem 10

Lemma. *If $p(x)$ is a polynomial, and m an integer, then*

$$\lim_{x\to 0^+} \frac{p(x)}{x^m} e^{-\frac{1}{x}} = 0.$$

Proof. As we are showing the limit is zero, we may show that

$$\lim_{x\to 0^+} \left| \frac{p(x)}{x^m} e^{-\frac{1}{x}} \right| = 0.$$

Let n be the degree of p. Then there is a polynomial q such that $p(1/x) = q(x)x^{-n}$.

As $x \to 0^+$, we have $1/x \to \infty$, so

$$\begin{aligned} \lim_{x\to 0^+} \left| \frac{p(x)}{x^m} e^{-\frac{1}{x}} \right| &= \lim_{x\to\infty} \left| p(1/x) x^m e^{-x} \right| \\ &= \lim_{x\to\infty} \frac{|q(x)x^{m-n}|}{e^x} \\ &\le \lim_{x\to\infty} \frac{|q(x)x^{|m|+|n|}|}{e^x} \end{aligned}$$

But e^x grows faster than any polynomial, so the limit is zero. □

We now let

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{-\frac{1}{x}} & \text{if } x > 0 \end{cases}$$

We will show

Lemma. *For all integer $n \ge 0$*

- *f is n-fold differentiable.*
- *$f^{(n)}(x) = 0$.*
- *There is a polynomial p_n such that $f^{(n)}(x) = \dfrac{p_n(x)}{x^{2n}} e^{-\frac{1}{x}}$ for $x > 0$.*

Proof. Our proof proceeds by induction on n.

For $n = 0$, the claims are clear, where $p_n(x) = 1$.

Now suppose it is true for n.

It is clear that $f^{(n)}$ is differentiable for $x > 0$, so f is $(n+1)$-fold differentiable for $x > 0$.

For $x = 0$, we have

$$\begin{aligned}(f^{(n)})'(0) &= \lim_{h\to 0} \frac{f^{(n)}(h) - f^{(n)}(0)}{h} \\ &= \lim_{h\to 0} \frac{f^{(n)}(h)}{h} \\ &= \lim_{h\to 0} \frac{p_n(h)}{h^{2n+1}}\, e^{-\frac{1}{h}} \\ &= 0\end{aligned}$$

by the previous lemma. This shows that f is $(n+1)$-fold differentiable even at $x = 0$, and in fact it shows that $f^{(n+1)}(0) = 0$.

Finally, for $x > 0$, we compute

$$\begin{aligned}(f^{(n)})'(x) &= \frac{d}{dx}\left(\frac{p_n(x)}{x^{2n}} e^{-\frac{1}{x}}\right) \\ &= e^{-\frac{1}{x}} \frac{d}{dx}\left(\frac{p_n(x)}{x^{2n}}\right) + \frac{p_n(x)}{x^{2n}} \frac{d}{dx}\left(e^{-\frac{1}{x}}\right) \\ &= e^{-\frac{1}{x}}\left(\frac{x^{2n}p_n'(x) - 2nx^{2n-1}p_n(x)}{x^{4n}}\right) + \frac{p_n(x)}{x^{2n+2}} e^{-\frac{1}{x}} \\ &= e^{-\frac{1}{x}}\left[\left(\frac{x^2p_n'(x) - 2nxp_n(x)}{x^{2n+2}}\right) + \frac{p_n(x)}{x^{2n+2}}\right] \\ &= e^{-\frac{1}{x}}\left(\frac{x^2p_n'(x) + (1-2nx)p_n(x)}{x^{2n+2}}\right).\end{aligned}$$

Therefore, taking $p_{n+1}(x) = x^2p_n'(x) + (1-2nx)p_n(x)$, our result is true.

□

We have now proven that our f is infinitely differentiable on its domain, and that all of its derivatives are zero. Therefore, its Taylor series is zero.

However, the function is not zero, as seen by taking $x = 1$, for which we get $f(1) = 1/e$. This implies that the function is not analytic.